Determing Accuracy Of A Taylor Polynomial In An Interval

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Consider the problem:

> Construct a Taylor polynomial approximation that is accurate within $\frac{1}{2} \times 10^{-3}$, over the given interval, for the function: $f(x) = e^{-x}$ where $x \in [0, 1]$. Expand about the point $x_0 = 0$.

Also, recall Taylor's Theorem:[1]

Taylor's Theorem:
If the function f possesses continuous derivatives of orders $0, 1, 2, \ldots, (n+1)$ in a closed interval $I = [a, b]$, then for any c and x in I,

$$f(x) = \sum_{k=0}^{n} \frac{f^k(c)}{k!}(x-6)^k + E_{n+1} \tag{1}$$

where the error term E_{n+1} can be given in the form

$$E_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1} \tag{2}$$

Here ξ is a point that lies between c and x, and depends on both.

Our goal is to determine how many terms in a Taylor polynomial are required to meet the desired accuraty of $\frac{1}{2} \times 10^{-3}$. To do this, we must determine when Eq. (2) is greater than or equal to $\frac{1}{2} \times 10^{-3}$. Essentially,

[1] *Numerical Methods and Computing, Fifth Edition* by Ward Cheney and David Kincaid [pg. 22].

we want to find the maximum values for $|f^{(n+1)}(\xi)|$ and $|(x-c)^{(n+1)}|$ in our interval. Let's begin by examining the first few derivatives of e^{-x}.

$$\begin{array}{ll} f'(x) = -e^{-x} & f''(x) = e^{-x} \\ f^3(x) = -e^{-x} & f^4(x) = e^{-x} \end{array}$$

Of course, the derivative of e^x is e^x and the derivative of e^{-x} is just going to alternate between positive and negative values of e^{-x}. Thus, $e^{-1} = 0.367879$ and $e^{-0} = 1$ giving us the maximum value of $|f^{(n+1)}(\xi)|$ at $\xi = 0$. And so Eq. (2) becomes:

$$E_{n+1} = \frac{1}{(n+1)!}(x-c)^{n+1} \tag{3}$$

Now we must maximize $|(x-c)^{n+1}|$. In this problem it is simple. We are expanding about $x_0 = 0$ so $c = 0$ and x ranges from 0 to 1. So, it doesn't matter what the value of n is; the maximum value of $(x-c)^{n+1}$ is always going to be 1 on our interval. From Eq. (3) we have:

$$E_{n+1} = \frac{1}{(n+1)!}(1)^{n+1} = \frac{1}{(n+1)!} \tag{4}$$

Which gives us the equation:

$$\frac{1}{(n+1)!} \leq \frac{1}{2} \times 10^{-3}$$

Solving this equation by hand can be tedious. Using technology to solve this equation is a much more judicious way to solve this inequality. For example, on a TI-89 one could enter "$0 \rightarrow n$`:While` $\frac{1}{(n+1)!} > \frac{1}{2} * 10^{-3}$`:n+1` $\rightarrow$ `n:EndWhile:Disp n:DelVar n`". The n displayed at the end of the loop execution will be the number of terms necessary for a Taylor polynomial approximation of e^{-x} accurate to $\frac{1}{2} \times 10^{-3}$. In this case, $n = 6$ and $T_6(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \frac{x^6}{720}$. Thus:

$$\begin{aligned} T_6(0.5) &\approx 0.6065321181 \\ e^{-0.5} &\approx 0.6065306597 \end{aligned}$$